On embeddings of finite metric spaces in l_∞^n

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March 25, 2009

Abstract

We prove that for any given integer $c > 0$ any metric space on n points may be isometrically embedded into l_∞^{n-c} provided n is large enough.

Let (X, ρ) be a metric space on n points. Denote by $m(X)$ the minimal k such that X may be isometrically embedded in l_∞^k and by $m(n)$ the maximum value of $m(X)$ for all metric spaces X on n points. Denote also $\alpha(X) = n - m(X)$, $\alpha(n) = n - m(n)$. It is well known that $m(n) \leq n - 1$. For example, one may fix a point $x_0 \in X$ and realize the l_∞^{n-1} as a vector space of functions on X, which vanish in x_0, endorsed with max-norm. Then the map $x \to \rho(x, \cdot) - \rho(x_0, \cdot)$ defines isometric embedding of X into this space. It is proved by D. Wolfe that if $n \geq 4$, then $m(n) \leq n - 2$ [5]. Using Ramsey-type graphs with n vertices without 4-cycles and k-anticliques K. Ball has shown [3] that $m(n) \geq n - k$. He refered to Alon's [1] explicit construction of such graphs with $k = O(n^{3/4})$, while the Spencer's combinatorial argument [4] allows us to get $k = O(n^{2/3} \cdot \log n)$, which gives $\alpha(n) = O(n^{2/3} \ln n)$.

Our main result is the following

Theorem 1. $\lim \alpha(n) = +\infty$, *i.e. that given* $c > 0$, $m(n) \leq n - c$ *for large enough* n.

The idea is to use the well-known observation that if $Y \subset X$, then $m(X) \leq m(Y) + |Y \setminus X|$, hence $\alpha(X) \geq \alpha(Y)$. Then we may try to find appropriate subset in X using Ramsey theorem and work with this subset instead X.

We start with the following technical

Definition 1. *A finite metric space X is called* generic, *if the distances between its points are linearly independent over $\mathbb{Q}$.*

and the standard

Lemma 1. *If $m(X) \leq N$ for any generic metric space X on n points, then $m(n) \leq n$, i.e. the same inequality holds for any metric space X on n points.*

Proof. Fix arbitrary metric space (X, ρ) on n points. Our aim is to prove that $M(X) \leq N$. Fix $\varepsilon > 0$. Change a metric on X by adding to each distance some number from $[\varepsilon, 2\varepsilon]$ so that the new metric ρ_ε is generic. It is clearly possible: just change the distances step by step, and on each step you have only countably many forbidden changes. Then (X, ρ_ε) may be embedded isometrically in l_∞^N, without loss of generality a point $x_0 \in X$ maps to 0. Then let ε tend to 0, consider the convergent subsequence of embeddings and its limit is isometric embedding of (X, ρ) in l_∞^{n-1}. $\square$

Hereafter we consider only generic metric spaces.

Assume that X is embedded in l_∞^N. For any $i = 1, 2, \ldots, N$ consider the value of i-th coordinate as a function ϕ_i defined on X. Note that ϕ_i's are 1-Lipschitz functions (that is, $|\phi_i(x) - \phi_i(y)| \leq \rho(x, y)$), and for any $x, y \in X$ there exists i such that equality $|\phi_i(x) - \phi_i(y)| = \rho(x, y)$ holds. These two conditions mean nothing but that our map is isometry on X. For any 1-Lipschitz function f on X define its (oriented) graph, vertices of which are points of X, and edge $a \to b$ is drawn iff $f(a) - f(b) = \rho(a, b)$. Then we realize that $m(X)$ is the minimal number of graphs of 1-Lipschitz functions on X, which cover all the edges of complete (non-oriented) graph on X. Recall the following well-known

Lemma 2. *If $Y \subset X$, then any 1-Lipschitz function f on Y may be extended to 1-Lipschitz function on X*

Proof. It suffices to consider the case $|X \setminus Y| = 1$, $X = Y \cup \{x_0\}$ (and then use induction). Define $f(x_0)$ as $f(x_0) = \max_{y \in Y}(f(y) - \rho(y, x_0))$. Let this maximum achieve in $z \in Y$. By definition, f satisfies $f(x_0) - f(y) \geq -\rho(y, x_0)$ for any $y \in Y$. So, to check that f is 1-Lipschitz on X, we need to check that $f(x_0) - f(y) \leq \rho(y, x_0)$ for any $y \in Y$. We have $f(x_0) - f(y) = f(z) - f(y) - \rho(z, x_0) \leq \rho(z, y) - \rho(z, x_0) \leq \rho(y, x_0)$ and we are done. $\square$

So, if some graphs of 1-Lipschits functions of $Y \subset X$ cover complete graph on Y, we may extend these functions to X so that they are still 1-Lipschitz and additionally consider the functions $\rho(z, \cdot)$ for all $z \in X \setminus Y$. The graphs of these functions cover all the edges having at least one endpoint not in Y. Therefore we get $m(X) \le m(Y) + |X \setminus Y|$. hence if we find a subset $Y \subset X$ such that $m(Y) \le |Y| - c$, then $m(X) \le |X| - c = n - c$ aswell.

Let T be a tree on X. Orient its edges so that there is no path of length 2 $a \to b \to c$. There are two ways to do it, choose any. Then define a function on X so that $f(a) - f(b) = \rho(a, b)$ for any edge $a \to b$ of our graph. It may be done uniquely to adding a constant function. Again, choose any. We need the following straightforward

Lemma 3. *If T is a tree such that any two vertices are joined by a path of at most 4 edges, then such a function is 1-Lipschitz iff for any path $a - b - c - d$ in a (non-oriented) tree T we have $\rho(a, d) + \rho(b, c) \ge \rho(a, b) + \rho(c, d)$.*

Proof. Consider any $x, y \in X$ and check whether $|f(x) - f(y)| \le \rho(x, y)$. If x and y are joined by edge, clearly the equality holds. If they are joined by the path $x - z - y$, then $|f(x) - f(y)| = |\rho(x, z) - \rho(z, y)| \le \rho(x, y)$, which is still ok. If they are joined by the path $x - z - w - y$, then $|f(x) - f(y)| = |\rho(x, z) - \rho(z, w) + \rho(w, y)|$. Inequality $\rho(x, z) - \rho(z, w) + \rho(w, y) \ge -\rho(x, y)$ always holds and follows from triangle inequality. Inequality $\rho(x, z) - \rho(z, w) + \rho(w, y) \le \rho(x, y)$ is exactly the condition which we require in Lemma. Finally, if x and y are joined by the path $x - z - t - w - y$, then $f(x) - f(y) = \pm(\rho(x, z) - \rho(z, t) + \rho(t, w) - \rho(w, y))$. For checking, say, $\rho(x, z) - \rho(z, t) + \rho(t, w) - \rho(w, y) \le \rho(x, y)$ use $\rho(x, z) - \rho(z, t) + \rho(t, w) \le \rho(x, w)$, which is again the requirement of Lemma. $\square$

Analogous statement holds for all trees, but here we need only trees of diameter at most 4.

Now we are ready for applying Ramsey theorem.

It is convenient to think that the points of our space are reals $x_1 < x_2 < \cdots < x_n$. For any indexes $1 \le a < b < c < d \le n$ consider the four points x_a, x_b, x_c, x_d and the following sums: $R_1 = \rho(x_a, x_b) + \rho(x_c, x_d)$, $R_2 = \rho(x_a, x_c) + \rho(x_b, x_d)$, $R_3 = \rho(x_a, x_d) + \rho(x_b, x_c)$. These sums are different reals (since X is generic), and there are six possible ways to rearrange them. So, take six colours labelled by elements of the symmetric group S_3 and colour the quadruple (a, b, c, d) in depend on the arrangement of R_1, R_2, R_3 (for example, colour (a, b, c, d) in color 231, if $R_2 > R_3 > R_1$).

Fix positive integer k and note that if n is large enough, then by Ramsey theorem there exist k indexes such that all quadruples formed by these indexes have the same colour. We call the corresponding subspace of X monochromatic of corresponding colour.

So, it suffices to consider only monochromatic spaces. Call our space (X, ρ), $|X| = n$ again, it is now generic and monochromatic.

We have six cases, which correspond to six permutations of 1,2,3. The first thing which we note is that two of them are impossible if $n \geq 5$.

Lemma 4. *There are no monochromatic generic metric spaces on 5 points of colours 213 or 312.*

Proof. Assume that (X, ρ) is a generic metric space on 5 points, monochromatic of colour 213. Let $x_1 < x_2 < x_3 < x_4 < x_5$ be the vertices of X. Then

$$\rho(x_1, x_2) + \rho(x_3, x_4) > \rho(x_1, x_4) + \rho(x_2, x_3)$$
$$\rho(x_2, x_3) + \rho(x_4, x_5) > \rho(x_2, x_5) + \rho(x_3, x_4)$$
$$\rho(x_1, x_4) + \rho(x_2, x_5) > \rho(x_1, x_2) + \rho(x_4, x_5).$$

Sum up to get a contradiction. The case of colour 312 is analagous, just change the sign in 3 above inequalities.

□

Monochromatic spaces of other four colours and arbitrary cardinality do exist, so we have to consider four cases. We use different approaches in all the four cases.

But before passing to separate cases, make the following general note about trees of diameter at most 4, which are graphs of 1-Lipschitz function (call such trees *admissible*). Each such tree has a *center*, some vertices, joined by edge with a center, which we call *main vertices*, and other, *peripheric*, vertices, each of them is joined with one of main vertices.

Note that if $Y \subset X$ and T is admissible tree on Y, then T may be extended to an admissible tree on X with the same set of main vertices. Indeed, it suffices to consider $X = Y \cup \{x_0\}$. Join x_0 with such a main vertex a that $\rho(x_0, a) - \rho(a, o)$ takes minimal value (o is center of T). Moreover, this expending is unique and so the admissible tree is uniquely determined by its center and main vertices. Denote the admissible tree with the center o and admissible vertices $a_1, a_2, \ldots, a_m$ as $T(o; a_1, a_2, \ldots, a_m)$.

Note also that the covering a monochromatic space by graphs of 1-Lipschitz functions does depend only on the colour of our monochromatic space and does not depend on metric structure in any other way. So, we may forget about the metric structure and define a unique monochromatic structure of given colour on any finite set of reals.

Case 321.

Let's prove that k trees are enough to cover 321-monochromatic space X with $2k$ vertices. Set $X = \{-k, -(k-1), \ldots, -1, 1, \ldots, k\}$. For any $i = 1, 2, \ldots, k$ consider the tree $T(-i; -k, -(k-1), \ldots, -i-1, 1, 2, \ldots, i) = T(i; k, (k-1), \ldots, i+1, -1, -2, \ldots, -i)$. It's straightforward to check that these trees are admissible and cover X. Note that they all have indeed diameter 3, not 4.

Case 132.

We use the following

Lemma 5. *Let Y be a 132-monochromatic space, $|Y| = n$ and m admissible trees cover the complete graph on Y without some k edges. Then we may add 2 vertices (i.e. 2 reals) to Y and for new space X add 2 admissible trees so that now the trees (2 added and extensions of m old) cover complete graph on X without $k - 1$ edges.*

In other words, we add two additional vertices and two additional trees, but kill one edge.

Proof. Assume that the edge $a - b$, $a < b$, is not covered by our trees. Add two vertices $a + \varepsilon$, $b - \varepsilon$ and two trees $T(a; a + \varepsilon, b)$ and $T(b; a + \varepsilon, b - \varepsilon)$. The check is straightforward. $\square$

Then, using this Lemma, we start with $|Y| = c$, $m = 0$ and $k = c(c-1)/2$. Apply the Lemma k times and get X such that $|X| = c + 2k$ and X is covered by $2k$ trees, as desired.

Case 123.

We prove that for given integer $c > 0$ there exists large N such that the 123-monochromatic space on N vertices may be covered by $N - c$ admissible trees. It is exactly what we need.

Use induction on c. For $c = 0$ take $N = 1$.

Now we show that if N is good for c, then $2N + 3$ is good for $c + 1$. Take vertices $-1, 0, 1, \ldots, 2N + 1$. Take the trees $T(0; -1, 2)$, $T(1; 0, 2)$ and $T(i + N + 1; i, i + 1)$ for $i = 1, 2, \ldots, N$. So, we use $N + 2$ trees and it

is straightforward that they cover all edges having at least one endpoint in $\{-1, 0, 1, \ldots, N+1\}$. All the other edges may be covered by $N - c$ trees by induction proposition. So, we use $2N + 2 - c = (2N + 3) - (c + 1)$ trees to cover all the edges.

Case 231

We take a 231-monochromatic space on $4n+1$ vertices $\{0, 1, \ldots, 4n\}$ and cover it by $3n$ trees. It is enough for our purpose for $n > c$.

For $i = 1, 2, \ldots, 2n$ take trees $T(0; i, 4n + 1 - i)$. They cover all the edges except edges $a - b$ for $1 \leq a \leq 2n < b \leq 4n + 1 - a$. So, we have to cover the bipartite graph (with n vertices in each part) formed by such edges by n trees. Note that the tree $T(1; 2n + 1, 2n + 2, \ldots, 4n)$ covers all its edge from vertices $1, 2n, 2n+1, 4n$. After taking this tree we have analogous graph with $n - 1$ vertices in each part by $n - 1$ trees. This is made by repeating this procedure (or by induction).

So, all the cases are considered and the Theorem 1 is proved.

Remark 1. *Another natural question is to study the minimal $k(n)$ such that any metric space on n point may be embedded in some k-dimensional Banach space, not necessary in l_∞^k. Clearly, $k(n) \leq m(n)$, and we do not know, whether the equality always holds or not. The probably best known lower bound for $k(n)$ is $2n/3$ [2].*

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